MSc-I
Quantum
Mechanics.

UNIT-III rd

field Quantization

Classical Mechanics: describes a discrete system i.e a system of single particle or a system of finite no. of particles.

CM → discrete

A continueous system with an infinite no. of degree of freedom is described by classical field theory. → continuous

- In Q.M (quantumn Mechanics), the dynamical variable like the coordinate of a particle become operators ($\hat{P}$, $\hat{E}$). In field theory, the field plays a role similar to a coordinates.
- Relativistic Q.M gives a description of the dynamics of a relativistic particle. But the interpretetation of the -ve energy solⁿ of the relativistic Q.M equations necessiate the existence of sea of infinite no. of particles filling all the -ve energy states.
- The relativistic Q.M therefore has to be a theory of infinitely many particles.
- A quantized field describes a system of

The construction of many particle theory is called field quantization or 2nd quantization & the theory is called quantum field theory.

1> Lagrangian field Theory:

The Lagrangian field theory is based on lagrangian - Hamiltonian canonical formulation of classical mechanics.

Steps in these formulation are:

Step - I :

Choose a set of generalised coordinate $\{q_i\}$ for a system.

Step - II :

Set up a lagrangian function.

$$L = L(q, \dot{q}, t) \quad --(1)$$

where $\dot{q}_i = \frac{dq_i}{dt}$ are generalised velocities & t denote time.

Step - III :

The action integral S_{21} b/w time t_1 & t_2 is defined as:

$$S_{21} = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt \quad --(2)$$

The condition that the change $\delta S_{21} = 0$ corresponding to a variation δq_i in q_i subject to the constraint

$\delta q_i(t_1) = \delta q_i(t_2) = 0$, then leads to the Euler - Lagrange eqⁿs:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad --(3)$$

looks like newton

Step-IV:

legendre transformation Define a Hamiltonian fuⁿ H by:

$$H = H(q, p, t) = \sum p_i \dot{q}_i - L \quad -(4)$$

p_i = momentum conjugate to the coordinate q_i

& $p_i = \frac{\partial L}{\partial \dot{q}_i}$ --(5)

Principle of least action states near classical path, variation of action = 0 $\delta S = 0$.

Using in (3)

$$\frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i}$$

$$\Rightarrow \boxed{\dot{p}_i = \frac{\partial L}{\partial q_i}} \quad --(6)$$

* The Hamilton's canonical eqⁿs are →

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad --(7)$$

from this, the eqⁿ for motion for a general dynamical variable $F(q, p, t)$ is given by:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H] \quad --(8)$$

where Poisson bracket of F & H

$$[F, H] = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

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a) <u>Canonical Quantization (CQ):</u>

for a mechanical system of particle the no. of generalized coordinate is equal to the no. of independent degrees of freedom of the system & this no. is finite.

The Quantization of such system is done by the following procedure:

<u>CQ</u> <u>I step</u>:

Replace all dynamical variable by corresponding Hermitian operator.

<u>for exp</u>:

The Q.M operator corresponding to H is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$$

$\hat{x}$ & $\hat{p}$ are Hermitian operator.

<u>Case II</u> → Replace the poission bracket [P.B] by commutator bracket:

$$[\hat{F}, \hat{H}] = \hat{F}\hat{H} - \hat{H}\hat{F}$$

<u>Case III</u> → Establish algebric relation for the basic canonical operator $\hat{q}_i$ & $\hat{p}_i$.

Heisenberg commutation rules:

i.e $[\hat{q}_i, \hat{q}_k] = 0$, $[\hat{p}_i, \hat{p}_k] = 0$ &

$[\hat{q}_i, \hat{p}_k] = i\hbar\delta_{ik}$.

> Coordinates of the field:

field has infinite no. of degree of freedom.

A field is specified by its amplitude at all points of space. Moreover, the amplitude at different space points are independent of each-other.

Thus the amplitude $\psi(r,t)$ play the same role in the case of a field as the G.C (generalized coordinates) $q_i(t)$ in the case of a mechanical system (finite no. of degree of freedom)

The no. of degree of freedom represented by $\psi(r,t)$ is infinite.

c> Classical field Equations:

The concept of field was introduced in classical Physics to explain the action at a distance.

A classical field reveals itself by producing a force on a material object that happen to into space occupied by the field.

Eg:

Electric & Magnetic field.

Acc. to Maxwell, In order to describe completely the state of the field in a given region of space, it is necessary to specify the strength & dirn of both electric & magn fields at every points of the region.

The role of a field in a continuum at every point, is similar to the role of a coordinate for a particle or a system of particle;

So, therefore develop Lagrangian & Hamiltonian formalism for a field:

The dependence of field strength or amplitude ψ on r which is a continuous variable, necessitate two types of modifications in this case:

i> to introduce a Lagrangian density $\mathcal{L}$.

The Lagrangian L of the field would be an integral of $\mathcal{L}$ over space.

ii> The Lagrangian density $\mathcal{L}$ would have to be a fun not only of $\psi, \dot{\psi}, \& t$ but also $\nabla\psi$.

Thus,

$$\mathcal{L} = \mathcal{L}(\psi, \dot{\psi}, \nabla\psi, t) \quad --(8)$$

$$\& \quad L = \int_V \mathcal{L}\, d^3r \quad --(9)$$

& action $$S_{21} = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} dt \int_V \mathcal{L}(\psi, \nabla\psi, \dot{\psi}, t)\, d^3r \quad --(10)$$

V = normalization volume.

The change δS_{21}, corresponding to infinitesimal variation $\delta\psi$ in ψ with $\delta\psi(r, t_1) = \delta\psi(r, t_2) = 0$ is given by

$$\delta S_{a1} = \int_{t_1}^{t_2} dt \int_V \delta \mathcal{L}\, d^3r \quad \text{--- (11)}$$

Now,

$$\delta\mathcal{L}(\psi, \nabla\psi, \dot\psi, t) = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\delta(\nabla\psi) + \frac{\partial\mathcal{L}}{\partial\dot\psi}\delta\dot\psi \quad \text{--- (12)}$$

$$= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\nabla(\delta\psi) + \frac{\partial\mathcal{L}}{\partial\dot\psi}\frac{\partial}{\partial t}(\delta\psi) \quad \text{--- (13)}$$

Using (13) in eqn (11) we have →

$$\delta S_{a1} = \int_{t_1}^{t_2} dt \int_V d^3r \left[\underset{\mathrm{I}}{\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi} + \underset{\mathrm{II}}{\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\nabla(\delta\psi)} + \underset{\mathrm{III}}{\frac{\partial\mathcal{L}}{\partial\dot\psi}\frac{\partial}{\partial t}(\delta\psi)}\right] \quad \text{--- (14)}$$

Now, IInd term can be solved as →

$$\int_V \frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\nabla(\delta\psi)\, d^3r = \sum_{x,y,z} \iint_{yz}\left\{\int_x \underset{\mathrm{I}}{\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_x}}\, \underset{\mathrm{II}}{\frac{\partial}{\partial x}(\delta\psi)}\, dx\right\} dy\, dz$$

$$= \sum_{x,y,z}\left\{\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_x}\cdot\delta\psi\right] dy\,dz - \int_V \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_x}\delta\psi\, dx\right) dy\,dz$$

$$= \Big[\underset{0}{\underbrace{\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_x}\delta\psi}} - \int_V \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_x}\right)\delta\psi\, dx\Big] dy\,dz$$

Surface integral vanishes because ψ either vanishes at ∞ or satisfies periodic boundary conditions.

⇒ using summation

$$-\int_V \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_x}\right)\delta\psi\, d^3r - \int_V \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_y}\delta\psi\, d^3r$$

$$-\int_V \frac{\partial}{\partial z}\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)_z}\right)\delta\psi\, d^3r$$

$$= -\int_V \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}(\delta\psi)\, d^3r \quad \text{--- (15)}$$

Now →

IIIrd term :

$$\int_{t_1}^{t_2} \overset{\mathrm{I}}{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}} \overset{\mathrm{II}}{\frac{\partial}{\partial t}(\partial\psi)}\, dt = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \int_{t_1}^{t_2} \frac{\partial}{\partial t}(\partial\psi)\, dt - \int_{t_1}^{t_2} \left\{ \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \cdot \int \frac{\partial}{\partial t}(\partial\psi) dt \right\}$$

$$= \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta\psi \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \delta\psi\, dt$$

$$= 0 - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \delta\psi\, dt$$

$$\boxed{(\because \delta\psi(r, t_1) = \delta\psi(r, t_2) = 0)}$$

$$= - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \delta\psi\, dt \quad \text{--(16)}$$

Using eqn (15) & (16) eqn (14) becomes →

$$\delta S_{21} = \int_{t_1}^{t_2} dt \int_V d^3r \left[\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi - \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla\psi)}\delta\psi - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right)\delta\psi \right]$$

$$= \int_{t_1}^{t_2} dt \int_V d^3r \left[\frac{\partial \mathcal{L}}{\partial \psi} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla\psi)} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \right]\delta\psi \quad \text{--(17)}$$

Acc. to Hamilton's principle of least action.

$$\delta S_{21} = 0$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \psi} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla\psi)} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = 0} \quad \text{- (18)}$$

This eqn is called classical field eqn in terms of lagrangian density.

{ It is analogue of Euler langrange eqn $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_i}\right) - \frac{\partial L}{\partial \xi_i} = 0$ in Classical Mechanics

Similarity of eqn (18) with eqn (3) is more clear when it expressed in terms of Lagrangian (L) instead of lagrangian density ($\mathcal{L}$) which can be done with the help of functional derivatives: ✓

For a free field, $\mathcal{L}$ is a fun of ψ & its derivatives, whereas L is a fun of ψ & $\dot{\psi}$. The difference is that while the value of a fun at a point r depends on the value of its arguments or independent variable at that point, the value of a functional depends on the value of its argument over the whole region or range.

$$L = L[\psi(r,t), \dot{\psi}(r,t)] \quad --(19)$$

variation →

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \psi_i}\delta\psi_i + \frac{\partial L}{\partial \dot{\psi}_i}\delta\dot{\psi}_i\right) \quad -(20)$$

In the continuum lt, eqn (20) is written as →

$$\delta L = \sum_i \lim_{\delta V_i \to 0}\left(\frac{1}{\delta V_i}\frac{\partial L}{\partial \psi_i}\delta\psi_i + \frac{1}{\delta V_i}\frac{\partial L}{\partial \dot{\psi}_i}\delta\dot{\psi}_i\right)\delta V_i \quad -21(a)$$

$$= \int_V \left(\frac{\partial L}{\partial \psi}\delta\psi + \frac{\partial L}{\partial \dot{\psi}}\delta\dot{\psi}\right) d^3r \quad --21(b)$$

where →

$$\lim_{\delta v_i \to 0} \frac{1}{\delta v_i} \frac{\partial L}{\partial \psi_i} = \frac{\partial L}{\partial \psi} \quad \text{--- (21) c}$$

$$\lim_{\delta v_i \to 0} \frac{1}{\delta v_i} \frac{\partial L}{\partial \dot{\psi}_i} = \frac{\partial L}{\partial \dot{\psi}} \quad \text{--- (21 d)}$$

Here, δv_i are volume of cells into which the volume V is divided & ψ_i is value of $\psi(r,t)$ at the ith cell

The variation of ψ & $\dot{\psi}$ at each cell can be done independently so that →

$$\delta \psi = \delta_{ij} \delta \psi_i \quad \text{--- (22a)}$$

$$\delta \dot{\psi} = \delta_{ij} \delta \dot{\psi}_i \quad \text{--- (22 b)}$$

$\frac{\partial L}{\partial \psi}$ & $\frac{\partial L}{\partial \dot{\psi}}$ from eqn (21c) & (21d) are functional derivatives of L w.r.t ψ & $\dot{\psi}$, resp.

Now as we know that →

$$L = \int_V \mathcal{L} \, d^3 r$$

$$\Rightarrow \delta L = \int_V \delta \mathcal{L} \, d^3 r$$

Using eqn (13), (15) & (16) we get

⇒

$$\delta L = \int_V \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi - \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \psi)} \delta \psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta \dot{\psi} \right] d^3 r$$

$$= \int_V \left[\left(\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi - \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \psi)} \delta \psi \right) + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta \dot{\psi} \right] d^3 r$$

$$= \int_V \left[\left(\frac{\partial \mathcal{L}}{\partial \psi} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \psi)} \right) \delta \psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta \dot{\psi} \right] d^3 r$$

--- (23)

On comparing eqn 23 with eqn 21 (k), we get → (6)

$$\boxed{\frac{\partial L}{\partial \psi} = \frac{\partial \mathcal{L}}{\partial \psi} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \psi)}}$$

&

$$\boxed{\frac{\partial L}{\partial \dot{\psi}} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}} \quad \text{--- (24)}$$

Using these in eqn (18) we get classical field eqn →

$$\frac{\partial L}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = 0} \quad \text{--- (25)}$$

which is similar to euler lagrange eqn in classical Mechanics. ✓

* Hamiltonian Formulation:

The momentum P_i conjugate to the canonical coordinate ψ_i is defined as

$$P_i(t) = \frac{\partial L}{\partial \dot{\psi}_i} \quad \text{--- (26)}$$ ✓

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4 the Hamiltonian of the field is given as→

$$H(t) = \sum_{i \leftarrow \text{discrete}} p_i \dot{\varphi}_i - L$$

Going over to the continuum limit, where φ_i is the value of $\varphi(x,t)$ in ith cell.

Now using eqn 21(d)→

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \lim_{\delta V_i \to 0} \frac{1}{\delta V_i} \frac{\partial L}{\partial \dot{\varphi}_i}$$

& Using eqn (24) in this equation→

$$\frac{\partial L}{\partial \dot{\varphi}(x,t)} = \delta V_i \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x,t)}$$

$d^3x = \delta V$

$$\Rightarrow \quad p_i(t) = \frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial \mathcal{L}\, d^3x}{\partial \dot{\varphi}(x,t)} = \pi(x,t)\delta V \quad \text{—— (27)}$$

Since $L = \int \mathcal{L} d^3x$

$$\& \quad H(t) = \int_V [\pi(x,t)\, \dot{\varphi}(x,t) - \mathcal{L}(x,t)] d^3x$$

$$= \int_V \mathcal{H}(x,t)\, d^3x \quad \text{-- (28)}$$

where $\boxed{\mathcal{H}(x,t) = \pi(x,t)\dot{\varphi}(x,t) - \mathcal{L}(x,t)}$ — 28(a)

Hamiltonian density.

Also,

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x,t)} \quad \text{—— 29(a)}$$

Conjugate field

$$\Rightarrow \pi(r,t) = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot\varphi}\right) = \frac{\partial \mathcal{L}}{\partial \varphi(r,t)} \quad \text{--- } 29(b)$$

$\boxed{\text{Using eq}^n \text{ 25}}$

clearly H is fun of π & φ. ✓

from 28,

$$H = \int_V (\pi\dot\varphi - \mathcal{L})\, d^3r$$

$$\delta H = \int_V \delta(\pi\dot\varphi)\, d^3r - \int_V \delta\mathcal{L}(r,t)\, d^3r$$

$$= \int_V \{\delta(\pi)\dot\varphi + \underbrace{\pi\,\delta(\dot\varphi)}_{(=0)}\}\, d^3r - \int_V \left(\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot\varphi}\delta\dot\varphi}_{(=0)}\right) d^3r$$

$$\left(\frac{\partial L}{\partial \dot\varphi} = \frac{\partial \mathcal{L}}{\partial \dot\varphi} \text{ eq}^n \text{ 24}\right)$$

$$= \int_V (\delta\pi)\dot\varphi\, d^3r - \int_V \pi(r,t)\,\delta\varphi\, d^3r$$

(using eq^n 29(b))

$$\delta H = \int_V \{(\delta\pi)\dot\varphi - \pi(\delta\varphi)\}\, d^3r \quad \text{--- } (30)$$

$\Rightarrow H = H[\varphi, \pi]$ i.e functional dependence.

$$\delta H = \int_V \left(\frac{\partial H}{\partial \varphi}\delta\varphi + \frac{\partial H}{\partial \pi}\delta\pi\right) d^3r \quad \text{--- } (31) ✓$$

Using eq^n (24) for H, we get -

$$\frac{\partial H}{\partial \varphi} = \frac{\partial \mathcal{H}}{\partial \varphi} - \nabla\cdot\frac{\partial \mathcal{H}}{\partial(\nabla\varphi)} \quad \text{--- } 32(a) ✓$$

$$\frac{\partial H}{\partial \pi} = \frac{\partial \mathcal{H}}{\partial \pi} - \nabla\cdot\frac{\partial \mathcal{H}}{\partial(\nabla\pi)} \quad \text{--- } 32(b)$$

On comparing eqⁿ 30 & 31 →

$$\dot{\psi}(r,t) = \frac{\partial H}{\partial \pi} \quad , \quad -\dot{\pi} = \frac{\partial H}{\partial \psi} \quad \text{--- (33)}$$

If F is an arbitrary functional of ψ & π, then

$$\dot{F} = \frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H] \quad \text{-- 34(b)}$$

$$= \frac{\partial F}{\partial t} + \int_V \left(\frac{\partial F}{\partial \psi}\dot{\psi} + \frac{\partial F}{\partial \pi}\dot{\pi} \right) d^3r \quad \text{-- 34(a)}$$

P.B ⇒ $[F, H]_{P.B} = \int_V \left(\frac{\partial F}{\partial \psi}\frac{\partial H}{\partial \pi} - \frac{\partial F}{\partial \pi}\frac{\partial H}{\partial \psi} \right) d^3 r$

$$\Rightarrow \frac{\partial H}{\partial \pi} = \dot{\psi} \ \ \& \ \ \frac{\partial H}{\partial \psi} = -\dot{\pi}$$

Now, $\dfrac{\partial \psi(r,t)}{\partial \psi(r',t)} = \delta(r - r')$ $\quad \left[\begin{array}{l} 1, \text{ for } r = r' \\ 0 \quad r \neq r' \end{array} \right.$

||ly $\dfrac{\partial \pi(r,t)}{\partial \pi(r',t)} = \delta(r - r')$ --- (35)

1) $[\psi(r,t), H]_{P.B\ \psi,\pi} = \dfrac{\partial \psi}{\partial \psi}\dfrac{\partial H}{\partial \pi} - \dfrac{\partial \psi}{\partial \pi}\dfrac{\partial H}{\partial \psi}$

$$\boxed{[\psi, H]_{PB} = \frac{\partial H}{\partial \pi(r,t)} = \dot{\psi}}$$ Using eqⁿ (33)

2) $[\pi(r,t), H]_{PB} = \dfrac{\partial \pi}{\partial \psi}\dfrac{\partial H}{\partial \pi} - \dfrac{\partial \pi}{\partial \pi}\dfrac{\partial H}{\partial \psi}$

$$= 0 - \frac{\partial H}{\partial \psi} = \dot{\pi} \quad \text{-- Using (33)}$$

3) $[\psi(r,t), \pi(r',t)]_{PB\ \psi,\pi} = \dfrac{\partial \psi(r,t)}{\partial \psi(r,t)}\dfrac{\partial \pi(r',t)}{\partial \pi(r,t)} - \dfrac{\partial \pi(r',t)}{\partial \psi(r,t)}\dfrac{\partial \psi(r,t)}{\partial \pi(r,t)}$

$$= \frac{\partial \pi(r',t)}{\partial \pi(r,t)} - 0$$

$$\Rightarrow \boxed{[\psi(r,t), \pi(r',t)]_{PB} = \delta(r' - r) = \delta(r - r')}$$

Q: Write down free schrodinger field langrangian & obtain the eqn of motion.

OR

Describe sailent features of IInd quartization using schrodinger field as e.g.

Non-Relativistic Field:

The Canonical formulation ensure that the quantized field is lorentz covariant if the corresponding classical field is lorentz covariant.

The quantized field is not lorentz covariant if the corresponding classical field is not lorentz covariant.

Ex. of non-relativistic field is →

1> Schrodinger field:

The classical field eqn of which is:

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi - V\psi = 0 \quad --(1)$$

from the viewpoint of Q.M, eqn (1) is the quantized eqn of motion of an ensemble of particle of mass m moving an external field having potential V.

But here we look upon it as a classical field eqn.

2nd quantization leads to the appearance of the field as an assembly of non-interacting indistinguishable particles. which is analogous to the normal modes of oscillation of a system of coupled oscillators in classical mechanics).

In term of lagrangian density classical field eqn → (using eqn (13)) ($\mathcal{L}_S$ due to schrodinger field)

$$\frac{\partial \mathcal{L}_S}{\partial \psi} - \nabla \cdot \frac{\partial \mathcal{L}_S}{\partial(\nabla\psi)} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}_S}{\partial \dot{\psi}}\right) = 0 \quad \text{--- (2)}$$

where

$$\boxed{\mathcal{L}_S = i\hbar\, \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi - V\psi^*\psi} \quad \text{--- (3)}$$

lagrangian density.

Now,

$$\frac{\partial \mathcal{L}_S}{\partial \psi} = -V\psi^*$$

$$\frac{\partial \mathcal{L}_S}{\partial(\nabla\psi)} = -\frac{\hbar^2}{2m}\nabla\psi^*$$

$$\frac{\partial \mathcal{L}_S}{\partial \dot{\psi}} = i\hbar\psi^*$$

Substituting in eqn (2) →

$$-V\psi^* - \nabla\cdot\left(-\frac{\hbar^2}{2m}\nabla\psi^*\right) - \frac{\partial}{\partial t}(i\hbar\psi^*) = 0$$

$$\Rightarrow -V\psi^* + \frac{\hbar^2}{2m}\nabla^2\psi^* - i\hbar\frac{\partial\psi^*}{\partial t} = 0$$

$$\Rightarrow \frac{\hbar^2}{2m}\nabla^2\psi^* - i\hbar\frac{\partial\psi^*}{\partial t} - V\psi^* = 0$$

$$\Rightarrow -i\hbar\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi^* - V\psi^* = 0$$

Taking complex conjugate →

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi - V\psi = 0}$$

which is eqn (1) g i.e classical field equation. ✓

The conjugate field $\pi(r,t)$ is given by

$$\pi(r,t) = \frac{\partial \mathcal{L}_S}{\partial \dot{\psi}} \quad \text{(Using eqn 29(a))}$$

$$= i\hbar\, \psi^*(r,t) \quad \text{--- (4)} \quad \text{Using eqn (2)}$$

& the $\mathcal{H}$ (Hamiltonian density) & H (Hamiltonian) is given by →

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L}_S \quad \because \text{using eqn 28 a)}$$

Using eqn (3) & (4) →

$$\mathcal{H} = i\hbar\psi^*\dot{\psi} - \left[i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V\psi^*\psi\right] ✓$$

$$\mathcal{H} = \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi + V\psi^*\psi$$

$$\left[\because \pi = i\hbar\psi^* \Rightarrow \psi^* = \frac{\pi}{i\hbar}\right]$$

$$\mathcal{H} = \frac{-i\hbar \cdot i\hbar}{2m}\nabla\psi^*\cdot\nabla\psi + \frac{1}{i\hbar}V\pi\psi$$

$$= \frac{-i\hbar}{2m} \times i\hbar\, \nabla\left(\frac{\pi}{i\hbar}\right)\nabla\psi - \frac{i}{\hbar}V\pi\psi$$

$$\boxed{\mathcal{H} = \frac{-i\hbar}{2m}\nabla(\pi)\cdot\nabla\psi - \frac{i}{\hbar}V\pi\psi} \quad \text{--- (5)}$$

Hamiltonian density

$$H = \int_V \mathcal{H}\, d^3r$$

$$\boxed{H = \int_V \left(\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi\right) d^3r} \quad \text{--- (6)}$$

Hamiltonian

a> Quantization :

We expand $\hat{\psi}(r,t)$ in terms of a complete orthonormal set of $\{u_k(r)\}$:

$$\hat{\psi}(r,t) = \sum_k \hat{a}_k(t)\, u_k(r) \quad --(7)$$

(↑ fourier coefficient)

Also

$$\hat{\psi}^{+}(r,t) = \sum_k \hat{a}_k^{+}(t)\, u_k^{*}(r) = \frac{-i}{\hbar}\hat{\pi}(r,t) \quad --(8)$$

We choose the $\{u_k(r)\}$ to be the energy eigen functions of the Hamiltonian of a single particle in the field.

$$\hat{H}_p = \frac{-\hbar^2}{2m}\nabla^2 + V(r) \quad --(9)$$

with $\hat{H}_p u_k(r) = e_k u_k(r) \quad --(10)$

Here, V is assumed to be independent of t. Quantization is done by postulating suitable algebric relations for the operator $\hat{a}_k(t)$ & $\hat{a}_k^{+}(t)$.

b> System of Bosons :

The commutation relation for the fourier coefficient are →

$$[\hat{a}_k, \hat{a}_l^{+}] = \hat{\delta}_{kl} \quad -(11(a))$$

$$[\hat{a}_k, \hat{a}_l] = 0 = [\hat{a}_k^{+}, \hat{a}_l^{+}] \quad -(11b)$$

here all the operator refer to the same time.

& we have an ∞ no. of operators a_k.
The eigenvalue spectrum of the Hermitian operator
$\hat{N}_k = \hat{a}_k^+ \hat{a}_k$ are non-negative integers.

$n_k = 0, +1, +2 - - \infty$ [∵ H.O]

A general state vector of the field is given by →

$$|n_1, n_2 - - n_k - - \rangle = c\,(\hat{a}_1^+)^{n_1} (\hat{a}_2^+)^{n_2} - - (\hat{a}_k^+)^{n_k} - |0\rangle \quad \text{--- (12(a))}$$

where $$C = \frac{1}{[n_1! n_2! - - n_k! - -]^{1/2}} \quad \text{--- (12b)}$$

& $|0\rangle$ is the vaccum state or G.S defined by

$$\hat{N}_k |0\rangle = 0 \quad \text{--- (12c)}$$

Also,

$$\hat{a}_k |n_1, n_2 - - n_k - - \rangle = \sqrt{n_k}\, |n_1, n_2 - (n_k - 1) - \rangle \quad \text{--- (13a)}$$

$$\hat{a}_k^+ |n_1, n_2 - n_k - - \rangle = \sqrt{n_k + 1}\, |n_1, n_2 - (n_k + 1) - \rangle \quad \text{--- (13b)}$$

Using eq^n (7) in eq^n (6), we get

$$\hat{H} = \sum_{k,l} \hat{a}_k^+ \hat{a}_l \int_V \left(\frac{\hbar^2}{2m} \nabla u_k^* \cdot \nabla u_l + V u_k^* u_l \right) d^3r \quad \text{--- (14a)}$$

$$= \sum_k \hat{N}_k \varepsilon_k \quad \text{--- (14b)}$$

The total energy of the field in a state $|n_1, n_2 - n_k - - \rangle$ is thus

$$E = \sum_k n_k \varepsilon_k \quad \text{--- (15)}$$

Here $\hat{a}_k^+$, $\hat{a}_k$, $\hat{N}_k$ are creation, annihilation, particle no. operators in the state u_k with energy ε_k.

The eigen vector or state vector $|n_1, n_2 --- n_k -- \rangle$ → gives an occupation no. representation for the system.

Since a given particle state u_k can be occupied by any no. of particle, the field represents an assembly of bosons.

c> System of Fermions:

We have seen that the quantized Postulates:

$$[\hat{a}_k, \hat{a}_l^+] = \hat{\delta}_{kl} \qquad (\text{if } k=l] \text{ then } [a, a^+] = 1 \quad [a^+, a] = -1$$

$$[\hat{a}_k, \hat{a}_l] = 0 = [\hat{a}_k^+, \hat{a}_l^+]$$

lead to a system of bosons.

For a system of fermions, the occupation no. n_k should be restricted to 0 & 1.

It has been shown by Jordan & Wigner that this condition could be met by replacing the commutation relations (11(a), 11(b)) by the following anticommutation relations:

$[N, a] = -a$ $\quad [N, a^+] = a^+$

$$\{\hat{a}_k, \hat{a}_l^+\} = \{\hat{a}_l^+, \hat{a}_k\} = \delta_{kl} \quad --(16a)$$

$$\{\hat{a}_k, \hat{a}_l\} = \hat{0} = \{\hat{a}_k^+, \hat{a}_l^+\} \quad --(16b)$$

Here also all the operators refers to the same time.

$\{a, a^+\} = 1$, $aa^+ + a^+a$, $aa^+ = 1-$

$$\hat{N}_k^2 = \hat{a}_k^+ \hat{a}_k \hat{a}_k^+ \hat{a}_k \qquad [= \hat{N}_k \cdot N_k]$$

$$= \hat{a}_k^+ \hat{a}_k - \hat{a}_k^+ \hat{a}_k^+ \hat{a}_k \hat{a}_k \qquad [\because \hat{a}_k \cdot \hat{a}_k = 0 = \hat{a}_k^+ \hat{a}_k^+]$$

$$= \hat{a}_k^+ (1 - \hat{a}_k^+ \hat{a}_k) \hat{a}_k \quad --(17)$$

$$N_k^2 = N_k$$

$$N_k N_k - N_k = \hat{0} \Rightarrow \hat{N}_k (\hat{N}_k - 1) = \hat{0} \quad -(18)$$

from which it follows that the eigen value spectrum of $\hat{N}_k$ is given by →

$$n_k = 0, 1 \quad --(19)$$

$$|n_1, n_2 - n_k - \rangle = (\hat{a}_1^+)^{n_1} (\hat{a}_2^+)^{n_2} -- (\hat{a}_k^+)^{n_k} --|0\rangle \quad --(20a)$$

$$\hat{a}_k |n_1, n_2 - - n_k - \rangle = (-1)^{\delta_k} n_k |n_1, n_2 - (n_k - 1) - \rangle \quad --(20b)$$

$$\hat{a}_k^+ |n_1, n_2 - - n_k - - \rangle = (-1)^{\delta_k} (1 - n_k) |n_1, n_2 - (n_k + 1) - - \rangle \quad (20c)$$

$$\hat{N}_k |n_1, n_2 - - n_k - - \rangle = n_k |n_1, n_2 - . n_k \rangle \quad --(20d)$$

where $\delta_k = \sum_{r=1}^{k-1} n_r \quad --(20e)$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p_x \right), \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} p_x \right)$$

$e q^n$ (20b, 20c) show that an empty state can't be further emptied & a filled state can't be further filled.

In Matrices form operators are:

$$[a_k] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad [a_k^+] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$[N_k] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The Hamiltonian $\hat{H}$ & the total energy E of the field are given in this case by ---(21)

$$\hat{H} = \sum_k \hat{N}_k \mathcal{E}_k \quad , E = \sum_k n_k \mathcal{E}_k$$

with n_k restricted to 0 or 1 (Pauli exclusion principle).

Also, in case of bosons as well as fermions, we can define an operator $\hat{N}$ representing the total no. of particles by

$$\hat{N} = \sum_k \hat{N}_k \quad \text{--(22)}$$

Now, $[\hat{N}, \hat{H}] = \sum_{k,l} [\hat{N}_k, \hat{N}_l] \mathcal{E}_l = 0$ —(23)

$\Rightarrow$ total no. of particles in the field is conserved.

for the case of bosons:

$$[\hat{\varphi}(r,t), \hat{\pi}(r',t)] = i\hbar\, \delta(r - \hat{r}') \quad \text{--(24a)}$$

commutator

$$[\hat{\varphi}(r,t), \varphi(r',t)] = 0 = [\pi(r,t), \pi(r',t)] \quad \text{—(24b)}$$

for fermions, the corresponding relations are 12

$$\{\hat{\varphi}(r,t), \hat{\pi}(r',t)\} = i\hbar\,\delta(r-r') \quad \text{---(25a)}$$

$$\{\hat{\varphi}(r,t), \hat{\varphi}(r',t)\} = \hat{0} = \{\hat{\pi}(r,t), \hat{\pi}(r',t)\} \quad \text{---(25b)}$$

The relations (24a) & (24b) resembles the Heisenberg commutation relations of quantum mechanics

$$\{\because [\hat{q}_i, \hat{q}_k] = \hat{0} = [\hat{P}_i, \hat{P}_k], \ [\hat{q}_i, \hat{P}_k] = i\hbar\,\hat{\delta}_{ik}\}$$

& thus could be regarded as the quantum theoretical extension of the classical relations & has no classical analogue.

Commutators & Anticommutators at Unequal times:

The Heisenberg eqn of motion for the field operator $\hat{\varphi}$ is given by

$$\frac{\partial\hat{\varphi}}{\partial t} = \dot{\varphi}(r,t) = \frac{1}{i\hbar}[\hat{\varphi}(r,t), \hat{H}] \quad \text{--(1)}$$

which is equivalent to →

$$\dot{\hat{a}}_k = \frac{d\hat{a}_k}{dt} = \frac{1}{i\hbar}[\hat{a}_k, \hat{H}]$$

$$= \frac{1}{i\hbar}\sum_l [\hat{a}_k, \hat{N}_l]\,\varepsilon_l$$

$$= \frac{1}{i\hbar}\hat{a}_k\,\varepsilon_k \quad \text{---(2)} \qquad [\because [\hat{a}_k, \hat{N}_l] = \delta_{kl}\hat{a}_k]$$

Thus, $\hat{a}_k(t) = \hat{a}_k(0)\exp[-(i/\hbar)\varepsilon_k t]$

& $\hat{a}_k^+(t) = \hat{a}_k^+(0)\exp[(i/\hbar)\varepsilon_k t]$

Now $[\hat{a}_k(t), \hat{a}_l^+(t')]_\pm = \delta_{kl}\exp[(i/\hbar)\varepsilon_k(t'-t)]$

$[\because [\hat{a}_k, \hat{a}_l^+] = \hat{\delta}_{kl}]$

$$[\hat{a}_k(t), \hat{a}_l(t')]_\pm = \hat{0} = [\hat{a}_k^+(t), \hat{a}_l^+(t)]_\pm$$

$$[\because [\hat{a}_k, \hat{a}_l] = \hat{0} = [\hat{a}_k^+, \hat{a}_l^+]]$$

where $[\hat{a}, \hat{b}]_+ \equiv \{\hat{a}, \hat{b}\}$, $[\hat{a}, \hat{b}]_- = [a, b]$

Quantization

- ① $\boxed{\text{Kelvin-Gordon field}}$
  for spinless particle (S=0) like π mesons & K mesons
  - $\boxed{\text{Real Scalar field}}$
    ↓ describes
    electrically Neutral Particles
    ↓
    Meson Propagator
    ↓
    $\boxed{\text{Meson exchange potential.}}$
  - $\boxed{\text{Complex Scalar field}}$
    ↓
    describes Charged Particles
- ② $\boxed{\text{Dirac field}}$
  for $S = 1/2$ Particles like fermions
  ↓
  fermion propagator
  ↓
  $\boxed{\text{compton Scattering}}$
  by electron
  by positron

Quantization of a real scalar field / Quantization of real Kelvin - Gordon field :

For a particle of rest mass m, energy & momentum are related by

$$E^2 = m^2c^4 + p^2c^2 \quad -(1) \text{ (relativistic expression)}$$

If the particles can be described by a single scalar wavefunⁿ $\phi(x)$, then K-G eqⁿ is given by →

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi + \left(\frac{mc}{\hbar}\right)^2 \phi = 0 \quad \text{(Unit II)}$$

$$\Rightarrow [\Box + \mu^2]\, \phi(x) = 0 \quad -(2)$$

where $\mu = mc/\hbar$

The interpretation of eqⁿ (2) as a single particle eqⁿ leads to difficulties.

∴ defining a positive definite particle density

i.e $P = \frac{E}{mc^2}(\phi\phi^*)$ (Unit II)

∴ Two signs of energy E resulting from eqⁿ (1)

$$\left[\because E = \pm\sqrt{m^2c^4 + p^2c^2}\right]$$

These difficulties are found in relativistic single - particle equations.

Such difficulties do not occur in many - particle theories which result when the fields, such as K-G field $\phi(x)$ are quantized.

Why K-G field is quantized ???

It is well known fact that a single scalar field posses orbitals but no spin angular momentum.

$\Rightarrow$ particles of spin 0.

Hence, the K-G eqn give the appropriate description of π - mesons (pions) & K-mesons, both of which have spin 0.

Let us consider a real scalar field $\phi(x)$ satisfying the K-G eqn (2).

Such a field corresponds to electrically neutral particles.

The K-G eqn (2) can be derived from the lagrangin density.

$$\mathcal{L} = \frac{1}{2}\left(\phi_\alpha \phi^\alpha - \mu^2 \phi^2\right) \quad -(3)$$

Here $\phi_\alpha \rightarrow$ covariant four vector & $\phi_\alpha = \frac{\partial \phi}{\partial x_\alpha} = \partial_\alpha \phi$

$\phi^\alpha \rightarrow$ contravariant four vector & $\phi^\alpha = \frac{\partial \phi}{\partial x^\alpha} = \partial^\alpha \phi$

$\mu = \frac{mc}{\hbar}$

$$\left\{\begin{array}{l} \phi_\alpha = \frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla\phi = \frac{\dot{\phi}}{c} + \nabla\phi \\ \phi^\alpha = \frac{1}{c}\frac{\partial \phi}{\partial t} - \nabla\phi = \frac{\dot{\phi}}{c} - \nabla\phi \end{array}\right\}$$

The four dimensional generalization of the gradient operator ∇ transforms like a four vector.

$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box = \partial^\mu \partial_\mu$

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$$\Rightarrow \mathcal{L} = \frac{1}{2}\left[\frac{\dot{\phi}^{2}}{c^2} - (\nabla\phi)^2 - \mu^2\phi^2\right]$$

The field conjugate to ϕ is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{c^2}\dot{\phi} \quad -(4)$$

On quantization the real field ϕ becomes a hermitian operator $(\phi^\dagger = \phi)$ & satisfying the commutation relations (at equal time) given by →

$$[\phi(x,t), \dot{\phi}(x',t)] = i\hbar c^2 \delta(x-x') \quad -(5a)$$

$$[\phi(x,t), \phi(x',t)] = [\dot{\phi}(x,t), \dot{\phi}(x',t)] = 0 \quad -(5b)$$

To establish contact with particles, we expand $\phi(x)$ in a complete set of solutions of the K-G eqn.

$$\phi(x) = \phi^+(x) + \phi^-(x) \quad -(6)$$

fourier expansion →

where,
$$\phi^+(x) = \sum_k \left[\frac{\hbar c^2}{2V\omega_k}\right]^{1/2} a(k) e^{-ikx} \quad -(6a)$$

↑ annihilation operator or absorption operator.

$$\phi^-(x) = \sum_k \left[\frac{\hbar c^2}{2V\omega_k}\right]^{1/2} a^+(k) e^{ikx} \quad (6b)$$

↑ creation operator

Now, wave vector
$$k = \frac{\omega_k}{c} = +\sqrt{\mu^2+k^2} \quad -(7)$$

& energy $E = \hbar\omega_k$

$$= \sqrt{m^2c^4 + p^2c^2} = \sqrt{m^2c^4 + c^2(\hbar k)^2} \quad -(8)$$

The commutation relations for the operators $a(k)$ & $a^+(k)$ are given by →

$$\left.\begin{aligned}[a(k), a^+(k')] &= \delta_{kk'} \\ [a(k), a(k')] &= [a^+(k), a^+(k')] = 0\end{aligned}\right\} \quad (8)$$

These are precisely the harmonic oscillator commutation relations.

The operators $N(k) = a_k^+ a_k$ have as their eigen values the occupation no.s

$$n(k) = 0, 1, 2 \ldots \quad (9)$$

The Hamiltonian & Momentum operators of the K-G field are given by

$$H = \int d^3x \cdot \frac{1}{2}\left[\frac{1}{c^2}\dot{\phi}^2 + (\nabla\phi)^2 + \mu^2\phi^2\right] \quad (10)$$

$$\because H = \int \mathcal{H}\, d^3x = \int d^3x\, (\pi(x)\dot{\phi}(x) - \mathcal{L}) \quad (11a)$$

$H = \Sigma \dot{x} p - L$

Using the eqn (4) $\boxed{\pi(x) = \frac{\dot{\phi}(x)}{c^2}} \Rightarrow \dot{\phi}(x) = \pi(x) c^2$

& $\boxed{\phi_\alpha = \frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla\phi}$ & $\boxed{\phi^\alpha = \frac{1}{c}\frac{\partial\phi}{\partial t} - \nabla\phi}$

Now from eqn (3)

$$\mathcal{L} = \frac{1}{2}\left[\phi_\alpha \phi^\alpha - \mu^2\phi^2\right]$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\left[\left(\frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla\phi\right)\left(\frac{1}{c}\frac{\partial\phi}{\partial t} - \nabla\phi\right) - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 - \mu^2\phi^2\right]$$

$$\mathcal{L} = \frac{1}{2}\left[\frac{\dot{\phi}^2}{c^2} - (\nabla\phi)^2 - \mu^2\phi^2\right]$$

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$$\Rightarrow H = \int d^3x \cdot \left\{ \frac{\dot\phi^2}{c^2} - \frac{1}{2}\frac{\dot\phi^2}{c^2} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 \right\}$$

$$= \int d^3x \cdot \left\{ \frac{1}{2}\frac{\dot\phi^2}{c^2} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 \right\}$$

$$H = \int d^3x \cdot \frac{1}{2} \left[\frac{\dot\phi^2}{c^2} + (\nabla\phi)^2 + \mu^2\phi^2 \right]$$

The momentum operator →

$$P = -\int d^3x \, \frac{\dot\phi}{c^2} \nabla\phi \quad \text{--- (11)}$$

$$\left\{ \dot p = -\frac{\partial H}{\partial \phi} = \frac{-\partial H}{\partial t}\frac{\partial t}{\partial \phi} \right\} \qquad \left(\begin{matrix} Q.M \\ \dot p = -\frac{\partial H}{\partial q} \end{matrix} \right)$$

Substituting the values from eqn (6) in eqn (10) & (11), we get →

$$H = \sum_k \hbar\omega_k \left(a^+(k)\, a(k) + \frac{1}{2} \right) \quad \text{--- (12)}$$

$$\& \; P = \sum_k \hbar k \left(a^+(k)\, a(k) + \tfrac{1}{2} \right) \quad \text{--- (13)} \quad \text{Constant of motion}$$

This confirms our interpretation of $[a^+(k) a(k)]$ as the no. operator for particles with wave number k.

From eqn (13), it is clear that momentum P is a constant of motion for the free K-G field.

The states of lowest energy, the g.s of the K-G field is the vaccum state $|0\rangle$ in which no particles are present i.e all $n(k) = 0$.

This state is represented by

$$a(k)\,|0\rangle = 0 \quad \forall k \quad \text{--- (14a)}$$

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& in term of field operators.

$$\phi^+(x)|0\rangle = 0 \quad \forall\, x \quad \text{---} (14b)$$

[∵ $\phi^+(x)$ depends upon $a(k)$. (eqn 6b)
& antihilation operator reduce no. of particle in any state by one. But in vaccum state there are no particle. so it can't reduce any more].

The vaccum has the ∞ energy $\frac{1}{2}\sum_k \hbar\omega_k$
So only observable is the energy diff. (being finite)
So all energies should be measured relative to vaccum state.

Using the relation $N(a(k_1)a(k_2)a^+(k_3)) =$

$$a^+(k_3)a(k_1)a(k_2) \quad \text{---} (15)$$

all absorption operators stands to right of creation operators.

$$N[\phi(x)\phi(y)] = N[(\phi^+(x)+\bar{\phi}(x))(\phi^+(y)+\bar{\phi}(y))]$$

(Using 6)

$$= N[\phi^+(x)\phi^+(y)] + N[\phi^+(x)\bar{\phi}(y)] + N[\bar{\phi}(x)\phi^+(y)] + N[\bar{\phi}(x)\bar{\phi}(y)]$$

$$= \underbrace{\phi^+(x)\phi^+(y)}_{\mathrm{I}} + \underbrace{\phi^-(y)\phi^+(x)}_{\mathrm{II}} + \underbrace{\phi^-(x)\phi^+(y)}_{\mathrm{III}} + \underbrace{\phi^-(x)\phi^-(y)}_{\mathrm{IV}} \quad \text{---} (16)$$

where order of the factors has been interchanged in the 2nd term i.e all +ve frequency parts ϕ^+ (which contain absorption operator $a(k)$) stand to right of all −ve frequency parts ϕ^- (which contain only creation operators $a^+(k)$

(16) (5) pl.

we redefine the Lagrangian density $\mathcal{L}$ & all observables (H, P) as their normal product.

Now, eqn (12) & (13) becomes →

$$P^{\alpha} = (H/c, P) = \sum_{k} \hbar k^{\alpha} a^{+}(k) a(k) \quad -(17)$$

we now construct one - particle state from the vaccum state $|0\rangle$ & is represented by

$$a^{+}(k)|0\rangle, \forall k \quad --(18)$$

& two particle state (k, k') is represented by

$$a^{+}(k) a^{+}(k')|0\rangle \quad \forall k \ \& \ k' \neq k \quad --(19a)$$

$$\& \ \frac{1}{\sqrt{2}}[a^{+}(k)]^{2}|0\rangle \quad \forall k \text{ i.e } k' = k \text{ also} \quad --(19b)$$

here factor $\frac{1}{\sqrt{2}}$ is the normalization factor.

[∵ normalization condition $\langle 0|0\rangle = 1$]

The particles of the K-G field are bosons so the occupation no. can take any value i.e $(n(k) = 0, 1, 2 --)$

eqn (19b) illustrates another aspect of boson state. they are symmetric under interchange of particle label.

[∵ on interchanging k by k', eqn (19a) remains same for two particle state]

Since all creation operators commute with each other, so we have,

$$a^{+}(k)\,a^{+}(k')|0\rangle = a^{+}(k')\,a^{+}(k)|0\rangle$$

Interchange

* Complex Scalar field:

Let us consider two real independent fields of identical mass m. The free lagrangian density is merely the sum of free lagrangian densities of the individual fields.

$$\mathcal{L}_{free}(\phi_1,\phi_2) = \mathcal{L}(\phi_1) + \mathcal{L}(\phi_2) = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}$$

$$+\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^{\,2} \quad --(1)$$

We construct complex field ϕ & ϕ^* given by

$$\phi = \left(\frac{\phi_1 + i\phi_2}{\sqrt{2}}\right), \quad \phi^* = \frac{(\phi_1 - i\phi_2)}{\sqrt{2}} \quad --(2).$$

Then $\phi_1 = \frac{1}{\sqrt{2}}(\phi + \phi^*)$, $\phi_2 = \frac{1}{\sqrt{2}\,i}(\phi - \phi^*)$ -- (3)

In terms of the complex field as defined in eq^n (2), the lagrangian density is →

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*) = \frac{1}{2}\phi^*_{,\mu}(x)\,\phi_{,}^{\;\mu}(x) -$$

$$\frac{1}{2}m^2\phi^*\phi \quad --(4)$$

substitution of (4) in the field eq^n yields →

$$(\Box + m^2)\phi = 0 \quad --(5)$$

& its conjugate →

$$(\Box + m^2)\phi^* = 0 \quad --(6)$$

We obtain the field eqn from the variational principle by treating ϕ & ϕ^* as two independent fields. Then

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = 0 \Rightarrow (\Box + m^2)\phi = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow [\Box + m^2]\phi^* = 0 \quad --(7)$$

Physical Interpretation of a complex field:

Let us consider a unitary transformation:

$$\phi \to \phi' = e^{i\lambda}\phi$$

$$\phi^* \to \phi^{*\prime} = e^{-i\lambda}\phi \quad --(8)$$

free lagrangian is invariant under this transformation. ∴ from K.G. eqn we have →

$$\frac{\partial f^\mu}{\partial x^\mu} = 0$$

with $f^\mu = \frac{\partial \mathcal{L}}{\partial_\mu \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial_\mu \phi^*}\delta\phi^* = i e\left(\frac{\partial \phi^*}{\partial x^\mu}\phi - \phi^*\frac{\partial \phi}{\partial x^\mu}\right) \quad --(9)$

⇒ ∃ a conserved current f_μ associated with the complex field ϕ.

Under the substitution $\phi \leftrightarrow \phi^*$ f_μ changes sign,

$\Rightarrow$ j_μ is to be interpreted as the electric current density & if ϕ corresponds to a particle with charge e then ϕ^* is a field corresponding to a particle with charge $-e$.

Means accommodate a pair of particles & antiparticles.

first → annihilation of particle
IInd → creation of antiparticle

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